

Fermi's golden rule and exponential decay as a RG fixed point

Edwin Langmann* and Göran Lindblad†

Department of Theoretical Physics
Royal Institute of Technology
SE-106 91 Stockholm, Sweden

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Abstract

We discuss the decay of unstable states into a quasicontinuum using models of the effective Hamiltonian type. The goal is to show that exponential decay and the golden rule are exact in a suitable scaling limit, and that there is an associated renormalization group (RG) with these properties as a fixed point. The method is inspired by a limit theorem for infinitely divisible distributions in probability theory, where there is a RG with a Cauchy distribution, i.e. a Lorentz line shape, as a fixed point. Our method of solving for the spectrum is well known; it does not involve a perturbation expansion in the interaction, and needs no assumption of a weak interaction. We use random matrices for the interaction, and show that the ensemble fluctuations vanish in the scaling limit. Thus the limit is the same for every model in the ensemble with probability one.

1 Introduction

The standard textbook derivation of Fermi's golden rule starts from a perturbation expansion of the unitary evolution, keeping the lowest nontrivial order, and then sums over a dense set of final states, to get a decay or reaction rate for an unstable state

$$\Gamma = 2\pi\rho v^2 \tag{1}$$

Here v represents the (average) transition matrix element and ρ is the density of final states. (We will use the convention $\hbar = 1$ throughout.) The formula is essentially contained in Dirac [8], see also [23]. If the rate Γ is a constant there results an exponential decay of the occupation number $p(t) = \exp(-\Gamma t)p(0)$. The corresponding quantum amplitudes are the Fourier transforms of a Lorentz line shape function, see equation (5).

It has been known for a long time that there are models where the golden rule and the exponential decay can be obtained without a perturbation expansion. It is our goal to show that such results hold for a large ensemble of models, and are exact in a suitable limit which leaves Γ invariant. This paper treats only the mathematical aspects of the

*E-mail: langmann@kth.se

†E-mail: gli@kth.se

models, mainly using known methods. It does not attempt to justify their physical relevance, but there are numerous applications of such models to lineshape problems, quantum chemistry, and many other fields; we will give a few references later.

The approach we use here was inspired by an important theorem of probability theory. The Lorentz line shape function (5) is a Cauchy probability density (PD), an infinitely divisible stable distribution. It is the distribution of a properly scaled limit of an infinite sum of independent identically distributed random variables, see Feller's book [11]. For any initial distribution the limit depends only on two real parameters, (Γ, a) in (5). This is similar to the role of the normal distributions in the central limit theorem, but the limit is adapted to PDs with long tails.

A physicist would call the Cauchy PD a fixed point for a renormalization group (RG) of transformations on the space of PDs, see Sections 2 and 3. Our thesis is that this is also a good way to approach our problem. We will define an ensemble of Hamiltonians where the spectral density defining the decay is a closed expression (23) in the Hamiltonian. There is no need for a perturbation expansion! In a simple scaling limit this spectral density will converge to one of the Cauchy type, a limit that can also be interpreted as the fixed point of a RG.

Many attempts at a rigorous derivation of rate equations (and quantum Markov master equations) use the van Hove limit, where there is a scaling of the interaction matrix elements (v) and the time (t) as follows

$$v \rightarrow \lambda v, \quad t \rightarrow \lambda^{-2} t \quad (2)$$

where $\lambda \rightarrow 0$ [5, 6, 21, 22]. In this way the dimensionless quantity Γt is invariant, but after the scaling the decay is on the rescaled (slow) time. Before taking such a limit it is necessary to let the density of states (DOS) ρ be infinite, otherwise there would be no relaxation in the limit $\lambda \rightarrow 0$. Here we will instead use the scaling

$$v \rightarrow \lambda v, \quad \rho \rightarrow \lambda^{-2} \rho \quad (3)$$

where $\lambda \rightarrow 0$, Γ is invariant, and t does not scale. The name continuum, or statistical, limit is often used, but note that here all eigenstates are normalized and the spectrum is always discrete (a quasicontinuum). The dimensionless number

$$N_\Gamma := \rho \Gamma \quad (4)$$

scales as $N_\Gamma \rightarrow \lambda^{-2} N_\Gamma$, and has a natural interpretation as the number of states under the resonance width. It is also the number of matrix elements effectively involved in the transition, a natural large parameter for the limit (3). We could replace the small parameter λ by $N_\Gamma^{-1/2}$.

We will show that (3) is the relevant scaling both for the limit theorem for the Cauchy PD, and for the quantum models introduced in Section 4, and that the scaling can be interpreted as the iteration of a RG. In the scaling limit the models will have exponential relaxation, and the rate will be given by the golden rule (1) or a matrix generalization of this formula.

An outline of the contents is as follows.

We first discuss the probabilistic properties of the Cauchy PD in Sections 2 and 3. In Section 4 we introduce the models of the effective Hamiltonian type and the spectral density for the relaxation is found, without any kind of expansion or approximation. The

models are defined as an ensemble of random matrices; in the simplest case the ensemble is defined by just two variable (non-random) parameters ρ and v .

The simplest case, a single level decaying into a quasicontinuum, is covered in Section 5. For mathematical simplicity we perform the calculations for the case where the unperturbed spectrum lies in a finite energy interval Δ with energy range ΔE . We can easily calculate the ensemble averages and show that they have the desired properties in the scaling limit. When we let $\Delta \rightarrow \mathbb{R}$ the $\lambda = 0$ limit of the spectral density is a Lorentzian with relaxation rate Γ defined by (1).

In Section 6 the variance of the fluctuations around the ensemble averages are estimated and shown to converge to zero in the scaling limit. Using the Chebyshev inequality we then conclude that each model in the ensemble will, with probability one, have an exponential relaxation with rate Γ in this limit. We also get a measure of how good this is as an approximation for finite parameter values; this enhances the physical relevance of the models. In particular, if $N_\Gamma \gg 1$ and $\Gamma \ll \Delta E$, the exponential relaxation is still a good approximation. We also note that the statistical ensembles can be dispensed with and replaced by certain uniformity assumptions on the spectra and interaction matrix elements.

In Section 7 the relation between the scaling (3) and the RG for the matrix models is analyzed. In Section 8 the unitary invariance of the random matrix ensemble is used to show that the results will hold for more general forms of the interaction. This invariance also allows us to show how the properties of the Cauchy distribution under convolution are related to those of the spectral density. In Section 9 we sketch how the models can describe several decaying states, and the relation to Fano's theory of lineshape is pointed out. Section 10 spells out the limits to exponential decay posed by a finite spectral range or a discrete spectrum. Finally there are some conclusions and additional remarks in Section 11.

2 Properties of the Cauchy distribution

The Lorentz (Breit-Wigner) line shape functions form a family of Cauchy PDs on \mathbb{R} , with parameters $a \in \mathbb{R}$ and $\Gamma > 0$

$$f_C(x - a, \Gamma) = \frac{\Gamma}{2\pi} \frac{1}{(x - a)^2 + \Gamma^2/4} \quad (5)$$

$$\int_{\mathbb{R}} dx f_C(x, \Gamma) = 1$$

Note that the mean and variance are not defined! The distribution function for $a = 0$ is

$$F_C(y, \Gamma) = \int_{-\infty}^y dx f_C(x, \Gamma) = \frac{1}{2} + \frac{1}{\pi} \arctan \left[\frac{2y}{\Gamma} \right] \quad (6)$$

When $\Gamma \rightarrow 0$ the limit is a unit step function, $F_C = 0$ for $y < 0$, and $F_C = 1$ for $y > 0$. Of course $F_C(-\infty, \Gamma) = 0$ and $F_C(\infty, \Gamma) = 1$ for all $\Gamma \geq 0$. The functions (5) form a convolution semigroup

$$\int_{\mathbb{R}} dx f_C(x - a_1, \Gamma_1) f_C(y - a_2 - x, \Gamma_2) = f_C(y - a_1 - a_2, \Gamma_1 + \Gamma_2) \quad (7)$$

see [11] section 2.4. Thus, if two independent RVs X_1 and X_2 have Cauchy PDs with parameters Γ_1 and Γ_2 , then $X_1 + X_2$ has a Cauchy PD with parameter $\Gamma_1 + \Gamma_2$. For

N independent RVs $\{X_1, X_2, \dots, X_N\}$, all with the same Cauchy PD with parameter Γ , the sum is Cauchy with parameter $N\Gamma$ and the average is Cauchy with parameter Γ

$$\frac{1}{N}S_N = \frac{1}{N} \sum_{k=1}^N X_k \sim X_1 \quad (8)$$

where \sim indicates equality in distribution. The characteristic function (CF) is

$$\chi(t) := \int_{\mathbb{R}} dx f_C(x - a, \Gamma) \exp(-ixt) = \exp \left[-iat - \frac{1}{2} \Gamma |t| \right] \quad (9)$$

The convolution (7) corresponds to a multiplication of the CFs.

Let the RVs in (8) have a common distribution F . Even when F is not Cauchy the scaled sum (8) can converge to a limit

$$X_\infty := \lim_{N \rightarrow \infty} \frac{1}{N} S_N \quad (10)$$

with a Cauchy distribution. Sufficient conditions on F for convergence can be found in [11] Section 17.5, especially in the Concluding remark, involving an assumption on the “tails” of F . The simplest case is that the following two limits exist and are equal

$$\lim_{x \rightarrow \infty} x [1 - F(x)] = \lim_{x \rightarrow \infty} x F(-x) = \frac{\Gamma}{2\pi} \quad (11)$$

where we can already identify the Cauchy parameter Γ . We let χ_F denote the CF of F and define a sequence of centering constants (with a dimensional parameter γ)

$$\beta_k := \gamma \int_{\mathbb{R}} \sin(x/k\gamma) F(dx), \quad k \in \mathbb{Z}_+ \quad (12)$$

It follows from (11) that the following limit exists [11]

$$\lim_{n \rightarrow \infty} \left[\chi_F(t/n) e^{i\beta_n t} \right]^n = e^{-|t|\Gamma/2} \quad (13)$$

The statement for the distribution F corresponding to (13) is

$$F_\infty(x) := \lim_{n \rightarrow \infty} (*^n F)[n(x + \beta_n)] = F_C(x, \Gamma) \quad (14)$$

We note the similarity to the central limit theorem, but there the factor $1/N$ in (10) is replaced by $1/\sqrt{N}$. If F has zero mean and a finite second moment then there is a convergence to a normal distribution, see [11], Section 8.4. A RG version of the central limit theorem is described in [17]. On the other hand for an F with a finite second moment the limit (14) is a step function corresponding to a δ -function density.

3 An example

We now want to give an intuitive hint why the mathematical result in the previous section is connected to the quantum lineshape problem. As an example of a distribution which fulfills (11) we pick one which is obtained from a standard lowest order perturbation expansion. Consider an infinite set of unperturbed quantum states $\{|k\rangle, k \in \mathbb{Z}\}$, with energy eigenvalues $E_k = k\omega$. There is a perturbation V with nonzero matrix elements

$\langle 0|V|k\rangle = v, \forall k \neq 0$. The perturbed state $|\psi_0\rangle$ which converges to $|0\rangle$ as $v \rightarrow 0$ has the occupation numbers, to the order v^2

$$p_k := |\langle k|\psi_0\rangle|^2 = \frac{v^2}{k^2\omega^2} \quad (k \neq 0), \quad p_0 := 1 - 2 \sum_{k=1}^{\infty} p_k \quad (15)$$

where v^2 must be small enough to make $p_0 > 0$, i.e. $v^2 \leq 3\pi^{-2}\omega^2$. The distribution is a step function with steps at $x = k\omega$

$$F(x) = \sum_{k\omega < x} p_k$$

and centered, hence (12) is zero. We find that the first limit in (11) reads

$$\frac{v^2}{\omega^2} \lim_{x \rightarrow \infty} x \sum_{k\omega \geq x} \frac{1}{k^2} = \frac{v^2}{\omega^2} \lim_{x \rightarrow \infty} x \int_{y\omega \geq x} \frac{dy}{y^2} = \frac{v^2}{\omega} = \rho v^2 = \frac{\Gamma}{2\pi}$$

and the second limit is identical; consequently we know that (14) holds with $\beta_n = 0$.

Note that $F * F$ has steps separated by ω , while $(F * F)[2x]$ has steps separated by $\omega/2$, and $(*^n F)(nx)$ in (14) has steps separated by ω/n . On the other hand we know from (7) if F is Cauchy then this convolution and rescaling recovers F . Consequently, this is an example of the scaling (3), when we identify Γ with (1).

As a preparation for later developments we consider a generalization to random distributions. Let V have random matrix elements $\langle 0|V|k\rangle = \xi_k$ where the RVs ξ_j are assumed to be complex-valued, normal, and independent, defined by the first two moments

$$\langle \xi_k \rangle = 0, \quad \langle \xi_j^* \xi_k \rangle = v^2 \delta_{jk} \quad (16)$$

The distribution function is now itself a random function. Consider the distribution of the tail function

$$T(x) := 1 - F(x) = \frac{1}{\omega^2} \sum_{k\omega \geq x} \frac{|\xi_k|^2}{k^2}$$

Clearly the ensemble average gives back what we had before

$$\lim_{x \rightarrow \infty} x \langle T(x) \rangle = \frac{\Gamma}{2\pi}$$

We can also calculate the ensemble variance. Introduce the real random vector

$$\eta_k := |\xi_k|^2 - v^2 \quad (17)$$

and use the standard properties of the normal RVs ξ to calculate

$$\langle \eta_k \rangle = 0, \quad \langle \eta_j \eta_k \rangle = \delta_{jk} v^4 \quad (18)$$

Define

$$\Delta T(x) := T(x) - \langle T(x) \rangle = \frac{1}{\omega^2} \sum_{k\omega \geq x} \frac{\eta_k}{k^2}$$

Then $\langle \Delta T(x) \rangle = 0$ and for large x

$$\langle \Delta T(x)^2 \rangle = \frac{1}{\omega^4} \sum_{k\omega \geq x} \frac{v^4}{k^4} \approx \frac{v^4}{\omega^4} \int_{y\omega \geq x} \frac{dy}{y^4} = \frac{v^4}{3\omega x^3}$$

and it follows that

$$\lim_{x \rightarrow \infty} x^2 \langle \Delta T(x)^2 \rangle = \lim_{x \rightarrow \infty} \frac{v^4}{3\omega x} = 0$$

Using the Chebyshev inequality we find as $x \rightarrow \infty$ (see [11], Chapter 5)

$$\text{Probability } \{|x\Delta T(x)| \geq \delta\} \leq \delta^{-2} x^2 \langle \Delta T(x)^2 \rangle \approx \frac{v^4}{3\delta^2\omega} \frac{1}{x} \rightarrow 0$$

We conclude that with probability 1 an element F in the ensemble will satisfy (11), consequently (13) and (14) will hold. However, the centering constants (12) can not be left out when the ξ_k have the distribution defined by (16).

4 Matrix models and resolvents

In this section we introduce the matrix models of the effective Hamiltonian type. For most of the discussion we assume the Hilbert space to be of finite dimension. In the limit $\lambda \rightarrow 0$ in (3) the dimension will be infinite, but again we emphasize that the eigenstates are all normalized. Also the limit of an unbounded discrete spectrum with finite DOS is interesting. We assume that these limits can be performed on the final results of the calculations without dealing too deeply with problems of mathematical rigor.

We note that the formulas derived in this section are known, the main ideas going back at least to Feshbach [12, 13, 18]. The mathematical background can be traced from Remark 2.1 of [16]. There are also textbook treatments with physical applications, see [3], Complements C_I and C_{III} , and [20], Chapter 21. However, we will use the results in a way which seems not to be standard.

Consider a self-adjoint matrix H (Hamiltonian), with discrete spectrum $\{\omega_\nu\}$, assumed non-degenerate for simplicity, and spectral projectors $\{P_\nu\}$

$$H := \sum_{\nu} \omega_{\nu} P_{\nu}$$

Define a causal resolvent (Green's function) with a regularization parameter $\varepsilon > 0$

$$R(z - i\varepsilon) := (z - i\varepsilon - H)^{-1} = \sum_{\nu} (z - i\varepsilon - \omega_{\nu})^{-1} P_{\nu}$$

We can pick a number of the projectors by integrating over a counterclockwise contour encircling the poles with real parts in a finite interval $I \subset \mathbb{R}$: $\{z = \omega_{\nu} + i\varepsilon, \omega_{\nu} \in I\}$

$$\sum_{\omega_{\nu} \in I} P_{\nu} = \frac{1}{2\pi i} \oint dz R(z - i\varepsilon)$$

For small ε we can approximate the contour integral by one along the real axis

$$\frac{1}{2\pi i} \int_I dx [R(x - i\varepsilon) - R(x + i\varepsilon)] = \frac{1}{\pi} \int_I dx \text{Im } R(x - i\varepsilon)$$

This means that we can consider the function

$$\frac{1}{\pi} \text{Im } R(x - i\varepsilon) = \frac{1}{\pi} \sum_{\nu} \frac{\varepsilon}{(x - \omega_{\nu})^2 + \varepsilon^2} P_{\nu} \quad (19)$$

as a regularized spectral density, normalized as follows

$$\frac{1}{\pi} \int_{\mathbb{R}} dx \operatorname{Im} R(x - i\varepsilon) = \sum_{\nu} P_{\nu} = \mathbb{1}$$

and the trace of (19) is a regularized DOS. The regularization can be written as a convolution (averaging) by a Cauchy PD

$$\operatorname{Im} R(x - i\varepsilon) = \lim_{\delta \rightarrow 0} \int_{\mathbb{R}} dy f_C(x - y, \varepsilon) \operatorname{Im} R(y - i\delta)$$

The regularization can be justified by noting that the duration of an observation of the system is bounded by a time scale $1/\varepsilon$, assumed much longer than the decay time.

Now write H as a block matrix, with projectors $\Pi_A, \Pi_B = \mathbb{1} - \Pi_A$ on the complementary subspaces

$$H = \begin{bmatrix} H_A & V \\ V^\dagger & H_B \end{bmatrix} \quad (20)$$

We know from the properties of the Schur complement [4] that

$$R_A(x - i\varepsilon) := \Pi_A R(x - i\varepsilon) \Pi_A = [x - i\varepsilon - \tilde{H}_A(x - i\varepsilon)]^{-1} \quad (21)$$

where, for any $z \in \mathbb{C}$, we define a non-Hermitian effective Hamiltonian

$$\tilde{H}_A(z) := H_A + V(z - H_B)^{-1} V^\dagger = \tilde{H}_A(z^*)^\dagger \quad (22)$$

We find for the imaginary part of (21), which, with a factor $1/\pi$, is the spectral density (19) projected on the subspace A

$$\begin{aligned} \phi_A(x, \varepsilon) &:= \frac{1}{\pi} \operatorname{Im} R_A(x - i\varepsilon) \\ &= \frac{1}{\pi} R_A(x - i\varepsilon) \left[\varepsilon + \operatorname{Im} \tilde{H}_A(x - i\varepsilon) \right] R_A(x - i\varepsilon)^\dagger \end{aligned} \quad (23)$$

For all $\varepsilon > 0$, $x \in \mathbb{R}$ this is a positive definite matrix. The FT is the matrix-valued CF

$$\chi_A(t, \varepsilon) := \int_{\mathbb{R}} dx \exp(-ixt) \phi_A(x, \varepsilon) \quad (24)$$

We also define the distribution function

$$\Phi_A(x, \varepsilon) := \int_{-\infty}^x dy \phi_A(y, \varepsilon) \quad (25)$$

Note that the normalization implies that for every ε ,

$$\Phi_A(-\infty, \varepsilon) = 0, \quad \Phi_A(\infty, \varepsilon) = \mathbb{1}_A$$

When $\varepsilon \rightarrow 0$ (25) converges to a step function $\Phi_A(x, 0)$; the steps are at $x = \omega_\nu$, the spectrum of H .

We can see in (23) the beginning of an exponential relaxation. In fact, if we could assume that $\tilde{H}_A(x - i\varepsilon)$ is independent of x , then (24) is a matrix-valued decaying amplitude

$$\chi_A(t, \varepsilon) = \exp \left(-i t \operatorname{Re} \tilde{H}_A - |t| \varepsilon - |t| \operatorname{Im} \tilde{H}_A \right) \quad (26)$$

Our goal is to justify this simple form in the limit (3), where we can finally set $\varepsilon = 0$. In most of the calculations below we will let the subspace A have rank 1, and then, if we leave out the argument $x - i\varepsilon$, the projected spectral density simplifies to

$$\phi_A(x, \varepsilon) = \frac{1}{\pi} \frac{\varepsilon + \text{Im} \tilde{H}_A(x - i\varepsilon)}{(x - \text{Re} \tilde{H}_A(x - i\varepsilon))^2 + (\varepsilon + \text{Im} \tilde{H}_A(x - i\varepsilon))^2} \quad (27)$$

Again, if $\tilde{H}_A(x - i\varepsilon)$ is constant we have the resonance form (5).

It is important for the success of this approach that the ε -averaged form of the resolvent (23) is expressed in terms of $\text{Im} \tilde{H}_A(x - i\varepsilon)$ which is also an ε -averaged expression, as we will see below.

5 The simplest model

The model (20) is particularly simple when the subspace indexed A has rank 1, and the spectral density has the form (27). This case has been solved repeatedly since a long time, and used for numerous applications, see for example [1, 3, 7, 10, 15]. Here the interaction V will represent an ensemble of random matrices. This is a device often used in applications [2, 19], here it allows us to make statements valid for almost all elements in the ensemble.

Now H_A is represented by single state $|\psi_s\rangle$, of energy E_s , while H_B is spanned by a finite set of states $\{|j\rangle; j = -N_B, -N_B + 1, \dots, N_B\}$. First we choose an equidistant spectrum $E_j = E_0 + j\omega_B$ where $E_0 \approx E_s$. The calculations will show that we can cope with more general spectra with an average DOS $\rho_B = 1/\omega_B$. We also assume that the energies are restricted to a finite interval Δ of width $\Delta E = E_+ - E_-$

$$E_j \in \Delta := [E_-, E_+]$$

Only a lower bound is essential for finite quantum systems, but here we prefer to simplify the mathematics by having a finite number of states in H_B , $N_B = 1 + \rho_B \Delta E$; later we can let this number approach ∞ .

The interaction V couples $|\psi_s\rangle$ with all the $|j\rangle$, allowing this state to decay. We choose the matrix elements to be independent complex normal random variables

$$\langle \psi_s | V | j \rangle = \langle j | V | \psi_s \rangle^* = \xi_j \quad (28)$$

defined by (16), while all other matrix elements are zero. Thus, in (20), $H_A = E_s$, V is a row vector with components ξ_j , and H_B is a diagonal matrix with elements E_j . The spectral density (27) is now

$$\frac{1}{\pi} \langle \psi_s | \text{Im} R(x - i\varepsilon) | \psi_s \rangle = \frac{\varepsilon}{\pi} \sum_{\nu} \frac{|\langle \psi_s | \omega_{\nu} \rangle|^2}{(x - \omega_{\nu})^2 + \varepsilon^2} \quad (29)$$

and in the limit $\varepsilon \rightarrow 0$ the FT (24) is the amplitude for staying in the state $|\psi_s\rangle$:

$$\langle \psi_s | \exp(-iHt) | \psi_s \rangle = \sum_{\nu} |\langle \psi_s | \omega_{\nu} \rangle|^2 \exp(-it\omega_{\nu})$$

For given E_s , H_B and V the equation system to solve for the eigenvalues ω_{ν} and eigenvectors $|\omega_{\nu}\rangle$ is [1]

$$\begin{aligned} (E_s - \omega) \langle \psi_s | \omega \rangle + \sum_j \xi_j \langle j | \omega \rangle &= 0 \\ (E_k - \omega) \langle k | \omega \rangle + \xi_k^* \langle \psi_s | \omega \rangle &= 0 \end{aligned}$$

We know that each eigenvalue ω_ν is located between two unperturbed ($V = 0$) eigenvalues, but we do not need the exact values. Instead we can estimate the terms in the RHS of (27). Decompose (22) into real and imaginary parts, for insertion in (27).

$$\tilde{H}_A(x - i\varepsilon) = E_s + \sum_j \frac{|\xi_j|^2(x - E_j)}{(x - E_j)^2 + \varepsilon^2} + i\varepsilon \sum_j \frac{|\xi_j|^2}{(x - E_j)^2 + \varepsilon^2} \quad (30)$$

We first deal with the imaginary part

$$\text{Im } \tilde{H}_A(x - i\varepsilon) = \varepsilon \sum_j \frac{|\xi_j|^2}{(x - E_j)^2 + \varepsilon^2} \quad (31)$$

A simpler case is solved e.g. in [1], where the interaction is non-random, $\xi_j = v$, and the spectrum of H_B is unbounded, i.e. $E_\pm = \pm\infty$. When $\omega_B \ll \varepsilon$ the infinite sum is approximated by an integral which is independent of the argument in \tilde{H}_A

$$\text{Im } \tilde{H}_A \approx \frac{\varepsilon v^2}{\omega_B} \int_{\mathbb{R}} dy \frac{1}{y^2 + \varepsilon^2} = \pi \rho_B v^2 = \frac{\Gamma}{2} \quad (32)$$

where we have put $\rho = \rho_B$ in (1). For our model, first apply the ensemble average over the RVs ξ_j , then, again, approximate by an integral

$$\begin{aligned} \langle \text{Im } \tilde{H}_A \rangle &= \varepsilon v^2 \sum_j \frac{1}{(x - E_j)^2 + \varepsilon^2} \approx \frac{\Gamma}{2} J(x, \varepsilon) \\ J(x, \varepsilon) &:= \frac{\varepsilon}{\pi} \int_{E_- - x}^{E_+ - x} dy \frac{1}{y^2 + \varepsilon^2} = \arctan \left[\frac{E_+ - x}{\varepsilon} \right] - \arctan \left[\frac{E_- - x}{\varepsilon} \right] \end{aligned} \quad (33)$$

It is easiest to see the behavior of the function J by a computer calculation. For $\varepsilon \ll \Delta E$ it is close to the CF $\chi(\Delta, x)$ for the interval Δ , with deviations ε -near the end points. When $\varepsilon \rightarrow 0$ then $J(x, \varepsilon) \rightarrow \chi(\Delta, x)$ and

$$\langle \text{Im } \tilde{H}_A(x - i\varepsilon) \rangle \approx J(x, \varepsilon) \frac{\Gamma}{2} \rightarrow \chi(\Delta, x) \frac{\Gamma}{2} \quad (34)$$

The same method applies to the real part

$$\begin{aligned} \langle \text{Re } \tilde{H}_A \rangle &= E_s + v^2 \sum_j \frac{(x - E_j)}{(x - E_j)^2 + \varepsilon^2} \approx E_s + \Gamma K(x, \varepsilon) \\ K(x, \varepsilon) &:= \frac{1}{2\pi} \int_{E_- - x}^{E_+ - x} dy \frac{y}{y^2 + \varepsilon^2} = \frac{1}{4\pi} \ln \left[\frac{(E_+ - x)^2 + \varepsilon^2}{(E_- - x)^2 + \varepsilon^2} \right] \end{aligned} \quad (35)$$

The second term in (35) represents an x -dependent level shift; it has a well-defined limit when $\varepsilon \rightarrow 0$. It has the effect of changing the DOS slightly, on the order of $\Gamma/\Delta E$. Near the center $\bar{x} = (E_+ + E_-)/2$ of Δ and for large ΔE

$$K(x, 0) = \frac{1}{2\pi} \ln \left[\frac{\Delta E - 2(x - \bar{x})}{\Delta E + 2(x - \bar{x})} \right] \approx -\frac{2}{\pi} \frac{x - \bar{x}}{\Delta E} \quad (36)$$

There is no unique way of taking the limit $\Delta E \rightarrow \infty$, but as long as x and \bar{x} stay bounded, it holds for all fixed $\varepsilon \geq 0$ that

$$\lim_{\Delta E \rightarrow \infty} K(x, \varepsilon) = 0 \quad (37)$$

Summing up the results so far: if we use the ensemble averages derived above for the terms in (27), and replacing the sums with integrals, we find the following approximation which we expect to be good for $\omega_B \ll \varepsilon$ (cf. [3], Complement C_{III})

$$\phi_A(x, \varepsilon) \approx \frac{1}{2\pi} \frac{2\varepsilon + \Gamma J(x, \varepsilon)}{[x - E_s - \Gamma K(x, \varepsilon)]^2 + [2\varepsilon + \Gamma J(x, \varepsilon)]^2/4} \quad (38)$$

6 Estimating the ensemble fluctuations

We want to justify the calculations in the previous section by showing that the ensemble fluctuations and other error terms vanish in the limit (3), thus that $\text{Im } \tilde{H}_A$ and $\text{Re } \tilde{H}_A$ are given by the last terms of (33) and (35), and that (38) becomes exact. We will concentrate on $\text{Im } \tilde{H}_A$, since the real part is very similar.

The deviation from the ensemble mean is given by

$$\Delta \text{Im } \tilde{H}_A := \text{Im } \tilde{H}_A - \langle \text{Im } \tilde{H}_A \rangle = \varepsilon \sum_j \frac{\eta_j}{(x - E_j)^2 + \varepsilon^2}$$

where η is the RV (17), with mean and variance given by (18). The variance of (31) is

$$\sigma^2(\text{Im } \tilde{H}_A) := \left\langle \left(\Delta \text{Im } \tilde{H}_A \right)^2 \right\rangle = \varepsilon^2 v^4 \sum_j [(x - E_j)^2 + \varepsilon^2]^{-2}$$

For an order of magnitude estimate we again replace the sum by an integral, and extend the integral to \mathbb{R} . Calculations similar to those performed above give

$$\sigma^2(\text{Im } \tilde{H}_A) \approx \frac{\pi v^4}{2\varepsilon \omega_B} = \frac{\omega_B \Gamma^2}{2\varepsilon 4\pi} \quad (39)$$

For the relative size of the fluctuations we can take the square root of this expression over (32), estimating the importance of the correction due to these fluctuations by the dimensionless parameter

$$\kappa := \sqrt{\frac{\omega_B}{2\pi\varepsilon}} = \frac{1}{\sqrt{2\pi N_\varepsilon}} \quad (40)$$

where $N_\varepsilon := \rho_B \varepsilon$. From (3) follows that it scales as $\kappa \rightarrow \lambda \kappa$ and goes to zero. Similar calculations show that κ also measures the fluctuations in $\text{Re } \tilde{H}_A$. We again use the Chebyshev inequality for an upper bound on the probability of having a deviation from the ensemble average [11]

$$\text{Probability}\{|\Delta \text{Im } \tilde{H}_A| \geq \delta\} \leq \frac{1}{2\pi N_\varepsilon} \left[\frac{\Gamma}{2\delta} \right]^2$$

Clearly, this quantity scales as λ^2 , and the scaling limit will give a value for $\text{Im } \tilde{H}_A$ and $\text{Re } \tilde{H}_A$ equal to the ensemble average.

We also have to deal with the error involved in replacing the sum over the spectrum by an integral, and in the assumptions on the level spacing. If the statistics for the level spacings form a Poisson process with parameter ρ_B , then the expected number of levels in an interval δE and the variance are equal

$$\langle N_{\delta E} \rangle = \rho_B \delta E = \sigma^2(N) := \langle (N_{\delta E} - \rho_B \delta E)^2 \rangle$$

Using the random version of the sequence $\{E_j\}$ in a sum like (30), we get the relative size of the resulting fluctuations by setting $\delta E = \varepsilon$

$$\frac{\sigma^2(N)}{\langle N \rangle^2} = \frac{1}{\rho_B \varepsilon} = \frac{\omega_B}{\varepsilon} \quad (41)$$

The square root is of the same order as κ (40). If instead we use a uniformly spaced spectrum, the integral approximation has a smaller error, of order (41) squared. We know that random matrix spectra are typically more uniform than the Poisson case; this is the feature of “spectral rigidity” due to level repulsion [19]. The correction term will then be in between the uniform and Poisson values. It appears reasonable to consider the Poisson value as a worst case for ensemble of spectra with a given density for the ensemble average.

The argument so far justifies (38) as an exact result in the limit (3). What happens when the limits $\varepsilon \rightarrow 0$ and $\Delta \rightarrow \mathbb{R}$ are taken after (3)? The order of taking the limits is not essential here. The limit $\varepsilon \rightarrow 0$ is straightforward in (38), and we get a normalized density $\phi_A(x, 0)$. This is a modified Cauchy density in a finite spectral interval and with a level shift function from (35) and (36). Finally, in the limit $\Delta \rightarrow \mathbb{R}$, using (37), we recover the exact Cauchy form

$$\lim_{\Delta \rightarrow \mathbb{R}} \phi_A(x, 0) = f_C(x - E_s, \Gamma) \quad (42)$$

In view of the fact that many applications involve models with large but finite DOS ρ_B , it is also interesting to see what we can say in this case, when we let $\varepsilon \rightarrow 0$ while $N_\Gamma \gg 1$. It is then convenient to use the integrated distribution function (25), which is always finite and monotonically increasing, while ϕ_A is a sum of δ -functions. Thus $\Phi_A(x, 0)$ is a step function, while the ε -averaged version $\Phi_A(x, \varepsilon)$ is close, as measured by the small number (40), to a Cauchy distribution (6) for $1 \ll N_\varepsilon \ll N_\Gamma$ and $\Gamma \ll \Delta E$. Given the properties of these two functions, the local averaging on an energy scale ε cannot have a drastic effect, and it must hold that

$$\Phi_A(x, 0) \approx F_C(x - E_s, \Gamma) \quad (43)$$

is a good approximation. This statement is supported by numerical calculations, see Figure 1. We can conclude that the distribution calculated from our model is close to Cauchy if there is a clear separation of energy scales

$$\omega_B \ll \Gamma \ll \Delta E \quad (44)$$

which also implies that $N_\Gamma \gg 1$.

To reach these conclusions, the properties of the spectrum of H_B and of the matrix elements of V were crucial. So far we have assumed simple regular or random distributions to do our estimates. Similar calculations are possible for more general sequences $\{E_j\}$ and $\{\xi_k\}$ without an assumed statistical distribution. Instead we can postulate a uniformity for the spectrum and the matrix elements, a property which could in principle be verified in a concrete model. For the level spacing assume that the following estimate holds uniformly in x , for all $\varepsilon \gg \omega_B$ and with κ defined by (40),

$$\frac{\omega_B \varepsilon}{\pi} \sum_j \frac{1}{(x - E_j)^2 + \varepsilon^2} = 1 + O(\kappa) \quad (45)$$

When $\varepsilon \rightarrow \infty$ (or $\varepsilon \rightarrow \Delta E$) the RHS must be essentially unity, and this fixes ω_B . In the same way the uniformity of the matrix elements means that

$$\frac{\omega_B \varepsilon}{\pi v^2} \sum_j \frac{|\xi_j|^2}{(x - E_j)^2 + \varepsilon^2} = 1 + O(\kappa) \quad (46)$$

where v^2 is defined through the limit $\varepsilon \rightarrow \infty$. The factor before the sum is $2\varepsilon/\Gamma$.

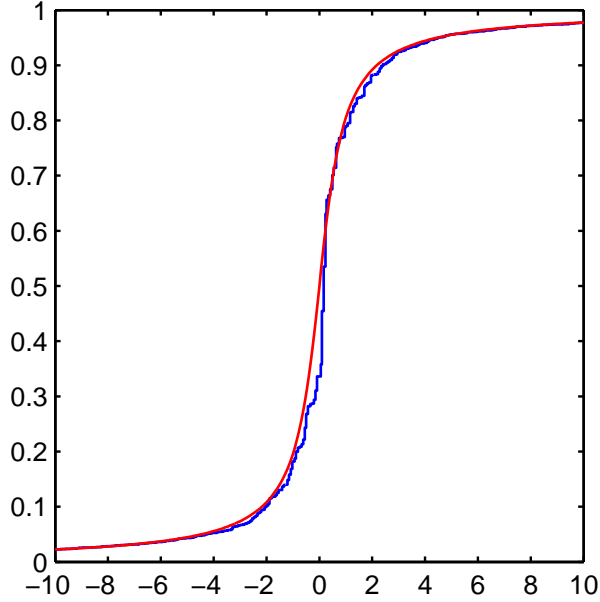


Figure 1: Illustration of the closeness of the computed distribution function (the step graph) to the Cauchy form given by (6) (the smooth graph) for a single random choice of the interaction V in the model of Section 5. The parameters are $N = 300$, $\Delta E = 20$, $\Gamma = 1.41$. Then $N_\Gamma = 21$, not a very large number in this context. For larger values we see a convergence to the Cauchy form.

7 Scaling and renormalization

It is easy to see how the scaling (3) works in the models of the type (20). If we disregard mathematical rigor the simplest form is obtained when the energy interval $\Delta = \mathbb{R}$. Then H in (20) transforms in a simple way: for $0 < \lambda < 1$

$$\begin{bmatrix} H_A & V \\ V^\dagger & H_B \end{bmatrix} \longrightarrow \begin{bmatrix} H_A & \lambda V \\ \lambda V^\dagger & \lambda^2 H_B \end{bmatrix} \quad (47)$$

Clearly these transformations form a semigroup, and we think of them as a RG. We can let H_B and V be random or not; it is enough that they fulfill the uniformity properties (45) and (46), which are preserved under the transformations. Of course, there is no proper limit for (47) as $\lambda \rightarrow 0$. For any initial choice of H of the type indicated above there a limit for Φ_A of the Cauchy form, and this is then the “fixed point” of the RG (47).

When $\Delta = \mathbb{R}$ we can also discuss the relation of the RG transformation (47) with that defining the Cauchy PD limit in Section 3. For large x the tail function of (25) is

$$T(x, \varepsilon) := 1 - \Phi_A(x, \varepsilon) \approx \frac{1}{\pi} \int_x^\infty dy y^{-2} \left[\varepsilon + \text{Im } \tilde{H}_A(y - i\varepsilon) \right]$$

For simplicity, again assume that the spectrum of H_B is $\{k\omega_B, k \in \mathbb{Z}\}$. Then a straightforward calculation gives

$$\lim_{\varepsilon \rightarrow 0} T(x, \varepsilon) = \frac{1}{\omega_B^2} \sum_{k\omega_B \geq x} \frac{|\xi_k|^2}{k^2}$$

Comparing with the calculations in Section 3 we again find the tail condition, with probability 1

$$\lim_{x \rightarrow \infty} x T(x, 0) = \frac{\Gamma}{2\pi}$$

Hence, setting $\Phi_A(x, 0) = F(x)$ we find that (13) and (14) hold. The two limiting procedures, that applied to the quantum models and that applied to PDs, give the same limits for an infinite uniform spectrum. The same result holds for a spectrum that is a stationary Poisson process.

We would like to define a RG transformation also when Δ is a finite interval. What does it look like in the matrix form (20)? We find a scaling (3) with $\lambda = 1/\sqrt{2}$, and which involves a doubling of the dimension of the matrices. To do this explicitly, consider a matrix composed of three parts, where the assumptions on H_A and the other components are as in Section 5

$$H = \begin{bmatrix} H_A & V_B & V_C \\ V_B^\dagger & H_B & 0 \\ V_C^\dagger & 0 & H_C \end{bmatrix} \quad (48)$$

We note the matrix identity

$$[V_B, V_C] \begin{bmatrix} H_B & 0 \\ 0 & H_C \end{bmatrix}^{-1} [V_B, V_C]^\dagger = V_B H_B^{-1} V_B^\dagger + V_C H_C^{-1} V_C^\dagger$$

and the corresponding decomposition of the effective Hamiltonian

$$\tilde{H}_A(z) = H_A + V_B(z - H_B)^{-1} V_B^\dagger + V_C(z - H_C)^{-1} V_C^\dagger$$

Thus the contributions of B and C to $\text{Im } \tilde{H}_A$ and to the decay of A just add up. Clearly $\rho_{B+C} = \rho_B + \rho_C$. In order deal as simply as possible with the energy level statistics, we can assume Poisson statistics for H_B and H_C with parameters ρ_B and ρ_C . For $B + C$ there will then be Poisson statistics with parameter ρ_{B+C} .

We can then let the $\lambda = 1/\sqrt{2}$ transformation be represented by

$$H_B \rightarrow H_{B+B'}, \quad V \rightarrow \frac{1}{\sqrt{2}} [V, V'] \quad (49)$$

where H_B and $H_{B'}$ have the same DOS ρ_B and independent Poisson statistics, while V and V' both have the distribution defined in (16). Clearly the number of energy levels scales as $N \rightarrow 2N$. We can choose for the RG an iteration of the dimension doubling, hence a sequence $\lambda_k = 2^{-k/2} \lambda_0 \rightarrow 0$, obtaining the limit discussed in Section 6, when we set $\varepsilon = 0$.

8 More general models

Here the solution of the model (20) in Section 5 is used to understand the properties of apparently more general models. We will also be able to see the relation between the convolution property (7) of the line shape function and the models defined here. Consider Hamiltonians of the form

$$H_1 = H_0 + V \quad (50)$$

where H_0 is a diagonal matrix with properties like those of H_B in (20). Again let E_k and $|k\rangle$ be the eigenvalues and eigenvectors of H_B , while the DOS is ρ_0 , and the spectral interval Δ . The eigenvalues and eigenvectors of H_1 are ω_ν and $|\omega_\nu\rangle$. The matrix elements

$$\langle j|V|k\rangle = \langle k|V|j\rangle^* = \xi_{jk}$$

are still assumed independent, identically distributed complex normal RVs

$$\langle \xi_{jk} \rangle = 0, \quad \langle \xi_{jk}^* \xi_{mn} \rangle = v^2 \delta_{jm} \delta_{kn} \quad (51)$$

We note the known fact that the ensemble of such random matrices V is invariant under all unitary transformations U , i.e. $U^\dagger V U \sim V$ [19]. Numerical simulations indicate that if $\Gamma := 2\pi\rho_0 v^2 \ll \Delta E$ and $\rho_0 \Gamma \gg 1$ then the DOS of H_1 is also ρ_0 , except near the endpoints of Δ .

Every eigenstate of H_0 will decay as a result of the interaction V . Numerical evidence indicates that the decay rate is near Γ when $\rho_0 \Gamma \gg 1$, and that the spectral densities are close to those found in Section 5. If this is true, then $|\langle k|\omega_\nu\rangle|^2$ inserted in (29) gives a spectral density near the Lorentz form, for every choice of k . (This fails for E_k near the end points of Δ , but we will argue as if $\Delta = \mathbb{R}$.) We now try to justify this picture, but without mathematical rigor.

Pick any eigenvalue of H_0 that is not too close to the ends of Δ . Call it E_0 and calculate how the corresponding eigenstate $|0\rangle$ decays by transforming this problem into that already solved in Section 5. Make a decomposition

$$H_1 = H_{1A} + H_{1B} + V_1 + V_1^\dagger$$

where there is only one non-zero element of $H_{1A} = E_0 + \xi_{00}$, V_1 is defined by $(V_1)_{0k} := \xi_{0k}$ and what is left is H_{1B} . For each ξ in the ensemble (51) we can diagonalize H_{1B} by a unitary U which leaves the basis vector $|0\rangle$ invariant. The transformed Hamiltonian is

$$U^\dagger H_1 U = H_{1A} + U^\dagger H_{1B} U + V_1 U + U^\dagger V_1^\dagger$$

which is now of the form (20), where $H_A \rightarrow H_{1A}$ is 1-D, and $U^\dagger H_{1B} U$ is diagonal with eigenvalues randomly distributed, but with a level repulsion which makes the DOS nearly uniform and equal to ρ_0 on the average. Due to the unitary invariance of the ensemble (51), it holds that $V_1 \sim V_1 U$. Thus the solution in Section 5 will still hold here for the decay of the chosen state, with ρ_B replaced by ρ_0 . The fluctuation properties will also be the same.

We make the following note on the invariance properties of the ensemble (51). The following phase transformation of the interaction part

$$V_{jk} \longrightarrow e^{i(\theta_j - \theta_k)} V_{jk}$$

leaves the ensemble invariant. It is easy to check that the eigenvalues are invariant and that the eigenvectors are just multiplied by a phase

$$\langle k|\omega_\nu\rangle \longrightarrow e^{-i\theta_k}\langle k|\omega_\nu\rangle$$

Now let $\zeta_k = \langle k|\zeta\rangle$ be the components of a vector. We can allow it to be a random vector, but assume it independent of the ensemble defined by (51). Then in

$$\langle \zeta|\omega_\nu\rangle = \sum_k \zeta_k \langle k|\omega_\nu\rangle$$

an average over the ensemble (51), or just over all angles θ_k , gives zero, while in the average of the absolute square the cross terms vanish

$$\langle |\langle \zeta|\omega_\nu\rangle|^2 \rangle = \sum_k |\zeta_k \langle k|\omega_\nu\rangle|^2$$

Using the spectral density form (29) we find for $\rho_0\Gamma \gg 1$

$$\langle |\langle \zeta|\omega_\nu\rangle|^2 \rangle \approx \rho_0^{-1} \sum_k \phi(\omega_\nu - E_k, \varepsilon) |\zeta_k|^2$$

and replacing ζ_k by a smooth function $\zeta(E_k)$ and approximating by an integral

$$\langle |\langle \zeta|\omega_\nu\rangle|^2 \rangle \approx \int_{\mathbb{R}} dx \phi(\omega_\nu - x, \varepsilon) |\zeta(x)|^2$$

i.e. a convolution.

We now add another interaction term with the same statistics

$$H_2 = H_1 + V' = H_0 + V + V'$$

i.e. $V \sim V'$, but we assume that V and V' are independent RVs. By the rules of adding independent normal RVs with zero mean $V + V' \sim \sqrt{2}V$. The diagonalization of H_2 can be done in one step using the interaction $V + V'$, or in two, first making H_1 diagonal, then H_2 , while using the unitary invariance of the ensembles. Use $|\Omega_u\rangle$ for the eigenvectors of H_2 . Then expand the scalar product

$$\langle k|\Omega_u\rangle = \sum_\nu \langle k|\omega_\nu\rangle \langle \omega_\nu|\Omega_u\rangle$$

take the absolute squared and a suitable ensemble average, as before

$$\langle |\langle k|\Omega_u\rangle|^2 \rangle = \sum_\nu |\langle k|\omega_\nu\rangle|^2 |\langle \omega_\nu|\Omega_u\rangle|^2$$

If we average over the angles only we need no averages in the RHS. Finally with ϕ_1 the spectral density coming from diagonalizing H_1 starting from H_0 , and ϕ_2 from diagonalizing H_2 starting from H_1 , we find the density for diagonalizing H_2 starting from H_0 as a convolution

$$\phi(E_k - \Omega_u, 2\varepsilon) \approx \int dx \phi_1(E_k - x, \varepsilon) \phi_2(x - \Omega_u, \varepsilon)$$

With the explicit resonance form (38), we find that ϕ_1 and ϕ_2 have parameter Γ while that ϕ in the LHS has parameter 2Γ , coming from $V + V' \sim \sqrt{2}V$. Thus we recover (7) in the case $\Gamma_1 = \Gamma_2 = \Gamma$.

9 Higher dimensions and Fano lineshapes

We return to the model (20) and allow H_A to have a finite dimension N_A . This kind of model can describe several states (interacting or not), all decaying into quasicontinua. In such applications we must make physical assumptions on the matrix elements of V , and it is not always relevant to assume them all independent as in (51). It is easy to use two different quasicontinua with different DOS, as in (48), perhaps coupled to two subset of levels in H_A .

Pick one possible structure by choosing ξ to be complex, normal RVs defined by

$$\langle \xi_{jr} \rangle = 0, \quad \langle \xi_{jr} \xi_{ks}^* \rangle = \gamma_{jk} \delta_{rs}$$

where γ is a positive semidefinite matrix of dimension $N_A \times N_A$. The imaginary part of (22) is then a positive semidefinite random matrix

$$G_{jk} := \left[\text{Im} \tilde{H}_A(x - i\varepsilon) \right]_{jk} = \varepsilon \sum_r \frac{\xi_{jr} \xi_{kr}^*}{(x - E_r)^2 + \varepsilon^2} \quad (52)$$

with ensemble average

$$\langle G_{jk} \rangle = \gamma_{jk} \varepsilon \sum_r \frac{1}{(x - E_r)^2 + \varepsilon^2} \approx \pi \rho_B \gamma_{jk}$$

Comparing with the case $N_A = 1$, it is natural to define the golden rule matrix

$$\Gamma_{jk} := 2\pi \rho_B \gamma_{jk} \quad (53)$$

The fluctuations around the average is estimated as in Section 6, and again found to contain a factor ω_B/ε , just like (39). Then (3) scales γ as $\gamma \rightarrow \lambda^2 \gamma$, and we will find the convergence to the ensemble mean in the limit $\lambda \rightarrow 0$ just as for $N_A = 1$. The same holds for $\text{Re} \tilde{H}_A$. Consequently the exponential solution (26) (including $\varepsilon = 0$) is justified.

When H_A and Γ commute and can be diagonalized simultaneously, each eigenstate decays independently and exponentially, each with a different Lorentz line shape and decay rate. When they do not commute, the present formalism includes models with Fano line shapes [7, 9, 10]. Assume that we can prepare the decaying system of N_A states in a definite quantum state $|\theta\rangle$. The lineshape function which will govern the non-exponential decay of this state is obtained from the spectral density (23) by projecting it on the corresponding 1-dimensional subspace (and setting $\varepsilon = 0$)

$$f(\theta, x) := \frac{1}{\pi} \langle \theta | \text{Im} R_A(x) | \theta \rangle$$

Clearly the factor R_A in (23) will in general have N_A poles in the upper half plane (and R_A^\dagger the conjugate poles in the lower half plane); these poles will determine the spectral density, which is far from the simple Lorentz form in general. Depending on the vector $|\theta\rangle$ there will be a weighted interference of the contributions from the resonance poles. The standard Fano lineshapes can be reproduced for $N_A = 2$ by suitable choices of the parameters. When $N_A > 2$ the number of different possibilities increase rapidly.

10 Separation of time scales

It is well known, for general models of decay, that if the energy spectrum of the reservoir is bounded then the decay will be non-exponential for very short and for extremely long time scales; this subject was discussed in e.g. [14]. For the class of models considered here the Lorentz line shape and exponential decay is an exact result in the limit (3) if we also set the energy interval $\Delta = \mathbb{R}$. We will very briefly analyze the deviations from this limit caused by a finite Δ or a finite ρ_B .

First consider the case when $\Delta = \mathbb{R}$ but $\rho_B < \infty$. When $\{E_k = k\omega_B; k \in \mathbb{Z}\}$, the “decay” is periodic with period $\tau_B := 2\pi\rho_B$. Clearly we need to have the decay time defined by Γ to be much shorter than that. This is so if $\omega_B \ll \Gamma$, or, equivalently $N_\Gamma \gg 1$. However, in order to find the time we need to see the deviation from an exponential decay to zero, another calculation is needed. Recall that $\Phi_A(x, 0)$ defined in (25) is a step function reflecting the discrete spectrum, but it still has a useful approximation (43). The decay (characteristic) function (24) will then be almost periodic. Consider the ergodic limit

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt |\chi_A(t)|^2$$

For exponentially decaying amplitudes this is zero, but for a discrete nondegenerate spectrum the limit picks out conjugate Fourier coefficients of $\chi_A(t)$ and $\chi_A(t)^*$ reducing the double sum to a single (diagonal) sum which is constant in time

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt |\chi_A(t)|^2 \approx \frac{\omega_B^2 \Gamma^2}{(2\pi)^2} \sum_k [(E_s - E_k)^2 + \Gamma^2/4]^{-2}$$

When the sum is approximated by an integral we find that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt |\chi_A(t)|^2 \propto \frac{\omega_B}{\Gamma} = \frac{1}{N_\Gamma}$$

Thus, over long times the function $|\chi_A(t)|^2$ is almost periodic, fluctuating around a non-zero mean value determined by N_Γ . We can expect to see the exponential decay in this fluctuating background only for t satisfying

$$t \Gamma < \ln N_\Gamma \tag{54}$$

Next we consider the effect of a finite ΔE while we can let $\rho_B = \infty$. Then $|\chi_A(t)|^2$ has a smooth quadratic maximum at $t = 0$. Numerical simulations for large values of ρ_B (taking into account the level shift) show that this function is close to an exponential for

$$t \Delta E \gg 1 \tag{55}$$

but they also indicate that there is a deviation from the exponential

$$|\chi(t)|^2 - \exp(-\Gamma t) = O(\Gamma/\Delta E)$$

which remains significant over a time interval of the order of $1/\Gamma$. When $\Gamma \ll \Delta E$ this deviation is very small, and the time $1/\Delta E$ is very short compared to the relaxation time scale. Then the effect of a finite ΔE on the relaxation will be insignificant, and the exponential is a good approximation.

11 Conclusions

In this paper we argue that the exponential relaxation and golden rule for model Hamiltonians of the type (20) can be understood as fixed point properties of this class under a simple RG of transformations, e.g. in the form (47).

The mathematical basis for the results is the simple non-perturbative form (23) for the spectral density and the assumed uniformity of the spectrum of H_B and the matrix elements of V , for instance in terms of the relations (45) and (46). In the random matrix approach the results hold with probability one for each element in the ensemble. The method works for a single decaying state, with a simple Lorentzian lineshape, as well as more complex cases, including Fano lineshapes, and it does not involve an expansion in the coupling strength. Instead we estimated the deviations from the scaling limit when the scaling parameter λ is small but nonzero, and found that the deviation is small if the number of states under the resonance is large: $N_\Gamma = \rho_B \Gamma \gg 1$. We could also handle the case of a finite energy range ΔE for H_B . Using the dimensionless quantities N_Γ and $N = \rho_B \Delta E$ (the total number of states) the condition for small deviations (44) reads

$$1 \ll N_\Gamma \ll N$$

In Section 10 we concluded that the exponential decay is a very good approximation in a time interval restricted by (54) and (55). In terms of N and N_Γ we found that

$$N_\Gamma/N \ll t\Gamma < \ln N_\Gamma$$

On the other hand, we note that these models cannot be used to derive the most general irreversible evolution for open quantum systems. The CF (26), acting on any pure initial state in the subspace A , defines a pure state, with decreasing norm, for all $t > 0$. This is clearly a special case, but still useful in many applications.

In view of the frequent use of the van Hove limit (2) it is important to say that (3) is not a weak coupling limit; instead it must be interpreted as a limit where the strength of the coupling is held constant while the reservoir is enlarged. In order to see this we first have to find a measure of the strength of the interaction term in the model of Section 5. When $N_B < \infty$ then we can choose the square root of the positive scalar (expectation for a random V)

$$\langle VV^\dagger \rangle = N_B v^2 \approx \rho_B v^2 \Delta E = \frac{1}{2\pi} \Gamma \Delta E$$

to get a measure of dimension energy. In the scaling (3) the parameters Γ and ΔE are invariant, so is the strength of the interaction. When $\Delta E \rightarrow \infty$ this value is inevitably misleading. The matrix elements connecting the decaying level to off-resonant final states, i.e. with $|E_s - E_k| \gg \Gamma$, will have little influence on the decay except for very short or very long times. Provided $\Gamma \ll \Delta E$ we can then replace ΔE by a multiple of Γ and use the scaling invariant Γ itself as a useful measure of the strength of the interaction.

Acknowledgments

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